On the Generalised Clark Haussmann Ocone Formula

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D.O.I: 10.56201/ijasmt.v8.no3.2022.pg86.91

Abstract
In this paper, exchange option and explicit computation of price of such option is examined. The variation of exchange option is investigated via the utilisation of the generalised Clark Haussmann Ocone Formula characterised by the Malliavin derivative. Thus, as an application, the European exchange price is explicitly computed thereby validating the efficiency of the Generalised Clark Hausmann Ocone Formula.

Keywords: *Generalised Clark Haussman Ocone Formula, Malliavin derivative, Martingale Representation Theorem, The Girsonov Theorem*
2010 MSC No:60H07

1 Introduction

The Exchange Option is the right but not the obligation an investor enjoys in exchange of one risky asset for another. The Exchange option was first introduced by Williams Margrabe as reported in [7] and its price was explicitly computed therein. The Generalised Clark Haussmann Ocone (CHO) formula is used in pricing a European Exchange Option as an application see [7]. The Clark- Haussmann-Ocone is a representation theorem of square integrable random variables in form of Ito-Stochastic integral, for which integrand is explicitly characterized in terms of the Malliavin derivative $\mathcal{D}_t$. In this paper, we study equation of the form

$$F = E[F] + \int_0^T E[\mathcal{D}_t F|F_t]dB_t \quad for \quad F \in \mathcal{D}_{1,2} \tag{1}$$

where $E[F]$ is the expectation of F, $\mathcal{D}_T$ is the Malliavin derivative. The equation (1) has the limitations that only the elements of the space $\mathcal{D}_{1,2}$. However, the goodnews is that the problem have been since circumvented to elements of the square interal functions $L^2(\Omega)$. This extension is possible through the application of the generalrised CHO formula via the Malliavin calculus route. Problems of this nature have been studied, see for instance [1,3,10] and the references therein. The power of the generalized Clark-Haussmann-Ocone (CHO) formula is that one can take advantage of the Malliavin derivative for computing the hedging portfolio.

2 Preliminaries

Definition 2.1 [10]

Suppose that $F \in \mathcal{D}_{1,2}$ such that there exists a sequence of random variables $\{F_n\} \subset \mathcal{P}$ with

$F_n \to F \in L^2(\omega)$ and $\{\mathcal{D}_t F_n\}_{n=1}^{\infty}$ converges to some functions in $L^2([0,T] \times \omega)$. Then, we can define $\mathcal{D}_t F = \lim_{n\to\infty}$ and

$$\mathcal{D}_\xi F = \int_0^T \mathcal{D}_t F . g(t)dt \quad g(s) \in L^2([0,T] \times \omega) \tag{2}$$

For all $\xi(t) = \int_0^T g(s)ds \ with g(s) \in L^2([0,T] \times \omega)$. Then, $\mathcal{D}_t F$ is called the Malliavin derivative.

Theorem 2.2[7]: The Martingale Representation Theorem

Let $\{B_t\}_{t\geq 0}$ be a Brownian motion with natural filtration $\{F_t\}_{t\geq 0}$ and let $\{X_t\}_{t\geq 0}$ be a square-integrable $(p, \{F_t\}_{t\geq 0})$ martingale i.e. $E^p[|X_t|^2 < \infty]$; $\forall t > 0$ Then the martingale representation theorem says that there is an adapted process $\{\theta_t\}_{t\geq 0}$ such that

$$X_t = X_0 + \int_0^t \theta_s dB_s \ , \quad a.s \tag{3}$$

$$(dX_t = \theta_t dB_t)$$

For the proof of this result, we refer to [7,10]

Theorem 2.3 [2]: The Girsanov Theorem

Let $Y(t) \in \mathcal{R}^n$ be an Ito-Process of the form

$$dY(t) = a(t,w)dt + dB(t); \quad t \leq T, \ Y_0 = 0 \tag{4}$$

where $T \leq \infty$ is a given constant and $B(t)$ is n-dimensional Brownian motion. For

$$M_t = \exp\left(-\int_0^t a(s,w)dBs - \frac{1}{2}\int_0^t a^2(s,w)ds\right); \quad t \leq T \tag{5}$$

Assume that $a(s,\omega)$ satisfies Novikov's condition

$$E\left[\exp\left(\frac{1}{2}\int_0^t a^2(s,w)ds\right)\right] < \infty \tag{6}$$

where $E = E_p$ is the expectation with respect to P. Define the measure Q on $(\Omega, F_t^{(n)})$ by

$$dQ(\omega) = M_t dp(\omega) \tag{7}$$

Then $Y(t)$ is a n-dimensional Brownian motion w.r.t the probability law Q, for $t \leq T$.

Investment [6,1]

Suppose that we are dealing with a complete market with two types of investment given as;

1. Safe investment: considering the first security as a risk-free asset, e.g bank account with price dynamics

$$dX_0(t) = p(t)X_0(t)dt \tag{8}$$

2. Risky investment (e.g. stocks) where for each $1 \leq i \leq n$, the price $X_i(t) = X_i(t,\omega)$ of stock number i is given by the Ito-diffusion

$$dX_i(t) = \alpha_i(t,\omega)dt + \sum_{j=1}^n \sigma_{ij}(t,\omega)dB_j(t) \tag{9}$$

$$X_i(0) = X_i$$

Let $\alpha(t) = \{\alpha_1(t,\omega), \cdots \alpha_n(t,\omega\}^T$, $\sigma(t)$ be the coefficients matrix of the volatility and the Brownian motion,

$$B(t) = B(t,\omega) = [B_1(t), B_2(t) \cdots . B_n(t)]^T$$

Then if

$$X(t) = \{X_1(t), X_2(t), \cdots X_n(t)\}^T$$

we have

$$dX(t) = \alpha(t)dt + \sigma(t)dB(t) \tag{10}$$

An investor who selects a portfolio consisting of $(n+1)$ assets will have to invest in each of the $(n+1)$ securities. The vector

$$\Theta(t) = \Theta(t,\omega) = \{\theta_0(t,\omega), \theta_1(t,\omega), \cdots, \theta_n(t,\omega)\} \tag{11}$$

represents the investor's holding at any time $t \in [0,T]$, where for each $1 \le i \le n$, $\theta_i(t,\omega)$ is the number of units of security number i that the investor will hold. In future, we shall refer to the vector of prices

$$X(t) = \{X_0(t), X_1(t), \cdots X_n(t)\}^T$$

as the market and the vector $\Theta(t,\omega)$ as the portfolio. In order to obtain the value of the portfolio,we denote $\eta(t)$ and $\gamma(t)$ as the capital invested at time t in the safe and risky investments respectively, then we can write the value of the portfolio $V^\Theta(t)$ as

$$V^\Theta(t) = \eta(t)X_0(t) + \gamma(t)X(t) \tag{12}$$

The portfolio is self financing if

$$dV^\Theta(t) = \eta(t)dX_0(t) + \gamma(t)dX(t) \tag{13}$$

$$dV^\Theta(t) = \rho(t)V^\Theta(t)dt + \gamma(t)\sigma\{\sigma^{-1}(\alpha(t) - \rho(t)X(t))dt + dB(t)\} \tag{14}$$

where $\gamma(t) = (\theta_1, \cdots, \theta_n)^T$. If we let $u = \sigma^{-1}(\alpha - \rho X(t)) = (u_1, \cdots, u_n)^T$, where $X(t)$ is the vector of stock prices and if we assume further that u satisfies the Novikov condition, then by the Girsanov theorem, $\bar{B}(t)$, $0 \le t \le T$ given by

$$d\bar{B}(t) = \sigma^{-1}(\alpha - \rho X(t))dt + dB(t)$$

$d\bar{B}(t) = udt + dB(t)$ is a standard Brownian vector with respect to the probability measure Q given by $dQ(w) = M(T)dp(\omega)$.

Therefore, equation (14) becomes,

$$dV^\Theta(t) = \rho(t)V^\Theta(t)dt + \gamma(t)\sigma d\bar{B}(t) \tag{15}$$

Solving for V^Θ, we get

$$e^{-\int_0^T \rho(s)ds}V^\Theta(t) = V^\Theta(0) + \int_0^T e^{-\int_0^t \rho(s)ds}\gamma(t)\sigma d\bar{B}(t) \tag{16}$$

$$e^{-\rho T}V^\Theta(t) = V^\Theta(0) + \int_0^T e^{-\rho t}\gamma(t)\sigma d\bar{B}(t) \tag{17}$$

3 The Gneneralised Clark Haussmann Ocone Formula

Thereom 3.1 The Generalised CHO formula [1]

Suppose that $F \in \mathcal{D}_{1,2}$ and assume that the following conditions hold

1. $E_Q\left[\| F \|_{L^2(Q)}\right] < \infty$
2. $E_Q\left[\int_0^T \| D_tF \|^2_{L^2(Q)} dt\right] < \infty$
3. $E_Q\left\{\| F \|_{L^2(Q)} \int_0^T \left(\int_0^T D_tu(s,w)dB(s) + \int_0^T D_tu(s,w) \cdot u(s,w)\right)^2 dt\right\} < \infty$

Then

$$F(\omega) = E_Q[F] + \int_0^T E_Q\left[D_tF - F\int_t^T (D_tu(s,\omega)d\bar{B}(s)|F_t)\right]d\bar{B}(t) \tag{18}$$

where $u(s,\omega)$ is the Girsanov kernel, Q is the equivalent Martingale measure and $\bar{B}(t)$ is a Brownian motion with respect to Q.

Remarks 3.2 [9]

By the uniqueness due to Martingale representation theorem, from (17) and (18), we have

$$V(0) = V^{\Theta}(0) = E_Q[F] \tag{19}$$

and

$$e^{-\rho t}\gamma(t)\sigma = E_Q\left[D_t G - G\int_0^T D_t u(s,\omega)d\bar{B}(s)|F_t\right] \tag{20}$$

where $\gamma(t) = (\theta_1, \cdots, \theta_n)^T$. Therefore the required risky investment at time t is

$$\gamma(t) = e^{-\rho(T-t)}\sigma^{-1}E_Q\left[D_t G - G\int_0^T D_t u(s,\omega)d\bar{B}(s)|F_t\right] \tag{21}$$

This gives the explicit number of units of stocks. The holding $\theta_0(t)$ in the bank account can be found from the self-financing condition.

The importance of these results is that in a complete market, every contingent claim with payoff $F(\omega)$ is attainable by a portfolio of stocks and bonds. Therefore $V(0)$, the initial value of self financing portfolio equals the price of a derivative, since

$$F(T) = V(T).$$

It then shows that the time zero price contingent claim is the discounted expectation of the payoff.

4 A European Exchange Option: An Application

4.1 The 2-Dimensional Market Model:[4, 6]

Suppose that the underlying securities in question have time t, prices $X_1(t)$ and $X_2(t)$ and to simplify our computations, we assume that the market consists of a bank account and the two stocks. The price of the bond is given by [5] albeit with constant force of interest ρ. The price of the risky securities are given by

$$X_i(t) = X_i(0)\exp\left[\left(\alpha_i - \frac{1}{2}\sum_{j=1}^2 \sigma_{ij}^2\right)t + \sum_{j=1}^2 \sigma_{ij}B_j(t)\right] \quad i = 1,2 \tag{22}$$

where, $B_j(t)$, $j = 1,2$ is a standard Brownian motion,such that the stochastic processes are defined on a filtered probability space (Ω, F, F_t, P) given that $F_t = F_t^2$ is a filtration for the assets such that the stochastic processes $\{X_i(t); t \geq 0\}$, $i = 1,2$ are adapted. Suppose that at terminal time $T > 0$, then

$$P[X_1(T) > X_2(T)] > 0.$$

Then the payoff of the exchange option will be

$$F(\omega) = [X_1(T) - X_2(T)]^+. \tag{23}$$

and the price X_i with respect to Q is

$$X_i(t) = X_i(0)\exp\left[\left(\rho - \frac{1}{2}\sum_{j=1}^2 \sigma_{ij}^2\right)t + \sum_{j=1}^2 \sigma_{ij}\bar{B}_j(t),\right] \tag{24}$$

where $\rho = [\alpha_1, \alpha_2]$ and $\bar{B}(t) = [\bar{B}_1(t) + \bar{B}_2(t)]^T$

In order to exploit the results from the previous discussion, we note here that the market $X(t) = (X_0(t), X_1(t), X_2(t))^T$ is a special case of the (n+1)-dimensional market considered in the previous section with n=2 in this case. We assume that σ is invertible so that the market is complete. Choosing a self-financing portfolio, yields

$$\Theta(t) = (\theta_0(t), \theta_1(t), \theta_2(t))^T \tag{25}$$

which is also admissible, then the discounted value of the portfolio at any time $0 \leq t \leq T$ is given by

$$e^{-\rho T}V^{\Theta}(t) = V^{\Theta}(0) + \int_0^t e^{-\rho s}\sigma\gamma(s)d\bar{B}(s) \tag{26}$$

where $\gamma(t) = (\theta_1(s), \theta_2(s))^T$. Therefore by uniqueness of the discounted value of the Portfolio and the Clark-Haussmann-Ocone (CHO) formula, due to the Martingale representation theorem, we have for any contingent claim $F(\omega) = V^{\Theta}(T)$, we get

$$V(0) = V^{\Theta}(0) = E_Q[e^{-\rho T}F(\omega)] \quad (27)$$
$$\gamma(t) = e^{-\rho(T-t)}\sigma^{-1}E_Q[D_tF(\omega)|F_t] \quad (28)$$

In order to facilitate our computation and taking advantage of the distribution of the terminal values of the underlying securities $X_1(t)$ and $X_2(t)$, we provide some important transformation results. Their usefulness will be evident in simplifying both (27) and (28).

Proposition 4.2 [8]

Let X_1 and X_2 be two independent m-dimensional normal random vectors each with mean equal to the zero vector and covariance matrix equal to the identity matrix and let $u \in \Re^m$ be any non-zero vector. Define a probability measure $P^{(u)} = Q$, equivalent to P with density

$$dP^{(u)}(\omega) = e^{uX_1 - \frac{1}{2}\|u\|^2}dP(\omega)$$

where $\|\cdot\|$ is the usual norm in $\Re^m$. Then $X_1 - u$ and X_2 are independent normal vectors with zero mean (vector) and covariance matrix equal to the identity.

Corollary 4.3 [4,5] Let X_1 and X_2 be two independent m-dimensional normal random vector as defined in the proposition above and let y_1 and y_2 be real numbers. If u_1 and u_2 are m-dimensional vectors, then

$$E_p[(S_1 - S_2)^+] = e^{y_1 + \frac{1}{2}\|u_1\|^2}\Phi\frac{y_1 - y_2 + \|u_1\|^2}{\sqrt{\|u_1\|^2 + \|u_2\|^2}} - e^{y_2 + \frac{1}{2}\|u_2\|^2}\Phi\frac{y_1 - y_2 - \|u_2\|^2}{\sqrt{\|u_1\|^2 + \|u_2\|^2}}$$

where, $S_1 = e^{y_1 + u_1X_1}$ and $S_2 = e^{y_2 + u_2X_2}$, $\|\cdot\|$ denotes the usual norm in $\Re^m$

As an application of the generalised CHO formula, the price of a European exchange option from equation (28) explicitly computed becomes,

$$V(0) = V^{\Theta}(0) = E_Q[e^{-\rho T}F(\omega)]$$

$$V(0) = X_1(0)\Phi(d_1) - X_2(0)\Phi(d_2) \quad (29)$$

where,

$$d_1 = \left[\frac{\ln\frac{X_1(0)}{X_2(0)} + \frac{T}{2}\sum_{j=1}^{2}(\sigma_{2j}^2 + \sigma_{1j}^2)}{\sqrt{T\sum_{j=1}^{2}(\sigma_{1j}^2 + \sigma_{2j}^2)}}\right] \quad and \quad d_2 = \left[\frac{\ln\frac{X_1(0)}{X_2(0)} - \frac{T}{2}\sum_{j=1}^{2}(\sigma_{2j}^2 + \sigma_{1j}^2)}{\sqrt{T\sum_{j=1}^{2}(\sigma_{1j}^2 + \sigma_{2j}^2)}}\right]$$

Therefore, following the above computation, we make the following remarks

Remarks 4.4

- The price does not depend on the appreciation rates of the stocks nor on the market interest rate ρ, but just on the market volatilities.
- The above result is similar to the one obtained in Margrabe (1978) but in that paper, the author considers the case when the Brownian motions are correlated and also with a special assumption that noise terms for each stock are different.
- The stock prices in this work depends on the two Brownian motions.

5 Conclusion

We have shown that the generalized Clark-Haussmann-Ocone (CHO) formula, becomes important in finding an expressions for the price of an exchange option. The price of a European exchange option has been explicitly computed from the Clark-Haussmann-Ocone (CHO) formula as the discounted expectation of the payoff with the two underlying stocks depending on the two Brownian motions. Unlike the Margrabe option which the stock prices were influenced each by one Brownian motion and the two were given as correlated. Extension of this work will be to obtain the hedging strategy from the integrant of the Clark Haussmann Ocone Formula, which is characterized by the Malliavin derivative.

ACKNOWLEDGEMENTS

The authors wish to acknowledge the contribution of our families.

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